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ON CAMERON-MARTIN THEOREM AND ALMOST SURE GLOBAL EXISTENCE

TADAHIRO OH AND JEREMY QUASTEL

ABSTRACT. In this note, we discuss various aspects of invariant measures for nonlinear Hamiltonian PDEs. In particular, we show almost sure global existence for some Hamiltonian PDEs with initial data of the form: “a smooth deterministic function + a rough random perturbation”, as a corollary to Cameron-Martin Theorem and known almost sure global existence results with respect to Gaussian measures on spaces of functions.

1. MAIN RESULTS

1.1. Introduction. In this note, we discuss almost sure global existence results for some nonlinear Hamiltonian partial differential equations (PDEs) as corollaries to Cameron-Martin Theorem [16]. In particular, we show almost sure global existence with initial data of the form

$$u_0(x; \omega) = v_0(x) + \phi(x; \omega), \quad (1.1)$$

where v_0 is a deterministic smooth function and $\phi(\omega)$ is a random function of low regularity. On \mathbb{T} , the function ϕ is given as¹

$$\phi(x; \omega) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{g_n(\omega)}{|n|^\alpha} e^{inx} \quad (1.2)$$

or

$$\phi(x; \omega) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle |n|^\alpha \rangle} e^{inx}, \quad \langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}, \quad (1.3)$$

where $\{g_n\}_{n \in \mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables on a probability space (Ω, \mathcal{F}, P) . For both (1.2) and (1.3), we easily see that ϕ lies almost surely in $H^{\alpha - \frac{1}{2} - \varepsilon}(\mathbb{T})$ for any $\varepsilon > 0$ but

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¹We drop the factor of 2π throughout the paper, when it plays no important role.

not in $H^{\alpha-\frac{1}{2}}(\mathbb{T})$. Note that (1.2) with $\alpha = 0$ corresponds to the mean-zero Gaussian white noise on \mathbb{T} :

$$\phi(x; \omega) = \sum_{n \in \mathbb{Z} \setminus \{0\}} g_n(\omega) e^{inx}. \quad (1.4)$$

Let us describe one of the motivations for studying the Cauchy problems with initial data of the form (1.1), namely

$$\text{a smooth deterministic function} + \text{a rough random perturbation.} \quad (1.5)$$

Given smooth physical data in an ideal situation, we may introduce rough and random perturbations to these data due to the limitations of accuracy in physical observations and storage of such data. Hence, we believe that it is important to study Cauchy problems with initial data of the form (1.5). Initial data (1.1) with (1.2) or (1.3) are the simplest models for (1.5) with rough Gaussian perturbations. One typical random noise we introduce in this kind of situation is the white noise, which appears ubiquitously in the physics literature. The white noise, however, is very rough and we can handle a smooth initial condition perturbed by the white noise only in a limited case.

1.2. Invariant Gibbs measures for Hamiltonian PDEs. Given a Hamiltonian flow on \mathbb{R}^{2n} :

$$\begin{cases} \dot{p}_j = \frac{\partial H}{\partial q_j} \\ \dot{q}_j = -\frac{\partial H}{\partial p_j} \end{cases} \quad (1.6)$$

with Hamiltonian $H(p, q) = H(p_1, \dots, p_n, q_1, \dots, q_n)$, Liouville's theorem states that the Lebesgue measure $\prod_{j=1}^n dp_j dq_j$ on \mathbb{R}^{2n} is invariant under the flow. Then, it follows from the conservation of the Hamiltonian H that the Gibbs measures $e^{-\beta H(p, q)} \prod_{j=1}^n dp_j dq_j$ are invariant under the dynamics of (1.6), where $\beta > 0$ is the reciprocal temperature.

In the context of the nonlinear Schrödinger equations (NLS) on \mathbb{T} :

$$iu_t - u_{xx} \pm |u|^{p-2}u = 0, \quad (x, t) \in \mathbb{T} \times \mathbb{R} \quad (1.7)$$

with the Hamiltonian:

$$H(u) = \frac{1}{2} \int_{\mathbb{T}} |u_x|^2 dx \pm \frac{1}{p} \int_{\mathbb{T}} |u|^p dx, \quad (1.8)$$

Lebowitz-Rose-Speer [29] considered the Gibbs measure of the form²:

$$\begin{aligned} d\mu &= d\mu^\beta = Z^{-1} e^{-\beta H(\phi)} d\phi \\ &= Z^{-1} e^{\mp \frac{\beta}{p} \int_{\mathbb{T}} |\phi|^p dx} e^{-\frac{\beta}{2} \int_{\mathbb{T}} |\phi_x|^2 dx} d\phi. \end{aligned} \quad (1.9)$$

²Throughout the paper, Z denotes various normalizing constants.

Here, $d\phi$ denotes the non-existent Lebesgue measure on the infinite dimensional phase space of functions on \mathbb{T} , and thus the expression (1.9) is merely formal at this point.

Noting that $e^{-\frac{\beta}{2} \int |\phi_x|^2 dx} d\phi$ is the Wiener measure on \mathbb{T} with variance β^{-1} , Lebowitz-Rose-Speer showed that such a Gibbs measure μ is a well-defined probability measure on $H^{\frac{1}{2}-\varepsilon}(\mathbb{T})$, $\varepsilon > 0$. In the focusing case, i.e. with the minus sign in (1.8), this construction holds only for $p < 6$ with the L^2 -cutoff $\mathbf{1}_{\{\|\phi\|_{L^2} \leq B\}}$ for any $B > 0$, and for $p = 6$ with sufficiently small $B > 0$.

Bourgain [3] continued this study and proved invariance of the Gibbs measure μ under the dynamics of NLS (1.7).³ See also McKean [30] for the cubic case. The main difficulty in [3] was to establish the global dynamics almost surely on the statistical ensemble. Bourgain achieved this goal by exploiting invariance of the finite dimensional Gibbs measures for the finite dimensional approximations to (1.7). In the same paper, he also considered the generalized KdV equations (gKdV):

$$u_t + u_{xxx} \mp u^{p-2} u_x = 0, \quad (x, t) \in \mathbb{T} \times \mathbb{R}. \quad (1.10)$$

with the Hamiltonian:

$$H(u) = \frac{1}{2} \int_{\mathbb{T}} u_x^2 dx \pm \frac{1}{p(p-1)} \int_{\mathbb{T}} u^p dx. \quad (1.11)$$

In particular, invariance of the Gibbs measures for KdV ($p = 3$) and mKdV ($p = 4$) was established in [3]. Recently, Richards [42] treated the case of the quartic KdV ($p = 5$). There have been papers in this direction by Bourgain [4, 6, 7, 8, 9] and other mathematicians that followed his idea [44, 45, 12, 14, 33, 36, 46, 31, 19, 20, 21].

In the following, we set $\beta = 1$ for simplicity. Then, the Gibbs measure μ in (1.9) is absolutely continuous with respect to the Wiener measure ρ with the density:

$$d\mu = Z^{-1} e^{-\frac{1}{2} \int_{\mathbb{T}} |\phi_x|^2 dx} d\phi. \quad (1.12)$$

A typical element ϕ in the support of the Wiener measure can be represented by the Fourier-Wiener series (1.2) with $\alpha = 1$.

In the defocusing case, the Gibbs measure μ and the Wiener measure ρ are equivalent, i.e. mutually absolutely continuous. In particular, almost sure global existence with respect to the Gibbs measure μ implies almost sure global existence with respect to the Wiener measure ρ . For example, the defocusing NLS (1.7) for any p is almost surely globally well-posed with

³In order to avoid the problem at the zero frequency, we need to insert $-\frac{\beta}{2} \int |u|^2 dx$ in (1.9) for NLS. As this is standard, we omit this term in the following for simplicity of the presentation.

respect to the random initial data $u|_{t=0} = \phi$, where ϕ is as in (1.2) with $\alpha = 1$. We point out that, for $p > 6$, this is beyond the known deterministic global well-posedness results. In Subsection 1.4, we show how this result can be extended to almost sure global well-posedness for the initial data $v_0 + \phi$, where $v_0 \in H^1(\mathbb{T})$ and ϕ is as in (1.2) with $\alpha = 1$.

Remark 1.1. In the following, we recall two properties of Gibbs measures. Although they are well known in probability theory and in statistical mechanics, we decided to include this remark for readers' convenience, in particular, for those in PDEs.

(i) Variational characterization of the Gibbs measure. Here, we restrict our attention to the finite dimensional setting (1.6). With $\phi = (p, q)$, the Gibbs measure can be written as

$$d\mu_\beta = f_\beta^*(\phi)d\phi := Z_\beta^{-1}e^{-\beta H(\phi)}d\phi, \quad (1.13)$$

where $d\phi$ denotes the Lebesgue measure $d\phi = \prod_{j=1}^n dp_j dq_j$ on \mathbb{R}^{2n} .

Given a probability measure ρ that is absolutely continuous with respect to the Lebesgue measure $d\phi$, we define its entropy $S(\rho)$ and average energy $\langle H \rangle(\rho)$ by

$$S(\rho) = - \int \frac{d\rho}{d\phi}(\phi) \log \left(\frac{d\rho}{d\phi}(\phi) \right) d\phi \quad \text{and} \quad \langle H \rangle(\rho) = \int H(\phi) \frac{d\rho}{d\phi}(\phi) d\phi,$$

respectively, where H is the Hamiltonian for the underlying dynamics. In the following, we consider the maximization problem of the entropy $S(\rho)$ for a given average energy $\langle H \rangle(\rho) = C$. We assume that, for a given value of C , there exists a unique $\beta > 0$ such that $\langle H \rangle(\mu_\beta) = C$. For simplicity of notations, we write $S(f)$ and $\langle H \rangle(f)$ for $S(\rho)$ and $\langle H \rangle(\rho)$, where $f := f_\rho = \frac{d\rho}{d\phi}$ denotes the Radon-Nikodym derivative of ρ with respect to the Lebesgue measure $d\phi$. Then, by the Lagrange multiplier method with two constraints $\langle H \rangle(f) = C$ and $M(f) := \int f(\phi)d\phi = 1$, we have

$$\begin{aligned} dS(f) &= \beta d\langle H \rangle(f) + \gamma dM(f) \\ \implies \int (\log f(\phi) + 1 + \gamma + \beta H(\phi))g(\phi)d\phi &= 0 \end{aligned}$$

for all test functions g . Thus, we conclude that $f(\phi) = e^{-1-\gamma-\beta H(\phi)}$. Moreover, by the mass constraint $M(f) = 1$, we must have $f(\phi) = Z_\beta^{-1}e^{-\beta H(\phi)} = f_\beta^*(\phi)$, where f_β^* is as in (1.13). Hence, if there is any extremal point for the entropy functional, it has to be the Gibbs measure μ_β . Also, by a direct computation, we have $d^2S(f)(g, g) = - \int \frac{g^2}{f} d\phi \leq 0$. Therefore, the Gibbs measure μ_β is the unique maximizer of the entropy for a given average energy.

(ii) Dependence of the Gibbs measure μ on $\beta > 0$: In the mathematics literature, the value of β is often set to be 1 for simplicity. In the following, we discuss the relation of μ^β for different values of $\beta > 0$. In particular, we show that the Gibbs measures μ^β and μ^γ , $\beta, \gamma > 0$, are singular if $\beta \neq \gamma$.

Consider the Gaussian measure ρ^β with the density:

$$d\rho^\beta = Z^{-1} e^{-\frac{\beta}{2} \int_{\mathbb{T}} |\phi_x|^2 dx} d\phi.$$

This is a Gaussian probability measure on $\dot{H}^s(\mathbb{T})$, $s < \frac{1}{2}$. Indeed, with $B_\beta := \beta^{-1} D^{2s-2}$, where $D = \sqrt{-\Delta}$, we have

$$-\frac{\beta}{2} \int_{\mathbb{T}} |\phi_x|^2 dx = -\frac{1}{2} \langle B_\beta^{-1} \phi, \phi \rangle_{\dot{H}^s}.$$

Hence, ρ_β is the (mean-zero) Gaussian measure with the covariance operator B_β . Moreover, with $e_n := |n|^{-s} e^{inx}$, $n \neq 0$, we have $B_\beta e_n = \lambda_n(\beta) e_n$, where

$$\lambda_n(\beta) = \beta^{-1} |n|^{2s-2}. \quad (1.14)$$

Now, consider two Gaussian measures ρ^β and ρ^γ , $\beta, \gamma > 0$. Feldman-Hájek theorem [22, 25] states that two Gaussian measures are either (i) equivalent or (ii) singular. Moreover, letting

$$S(\beta, \gamma) = \sum_{n \neq 0} \frac{(\lambda_n(\beta) - \lambda_n(\gamma))^2}{(\lambda_n(\beta) + \lambda_n(\gamma))^2},$$

we have (i) ρ^β and ρ^γ are equivalent if $S(\beta, \gamma) < \infty$ and (ii) ρ^β and ρ^γ are singular if $S(\beta, \gamma) = \infty$. See also Kakutani's dichotomy theorem [26] on equivalence of infinite product measures.

With (1.14), it is easy to see that $S(\beta, \gamma) = \infty$ if $\beta \neq \gamma$. Thus, ρ^β and ρ^γ are singular, if $\beta \neq \gamma$. Therefore, noting that μ^β is absolutely continuous with respect to ρ^β , it follows that the Gibbs measures μ^β and μ^γ are singular if $\beta \neq \gamma$.

Next, recall that NLS (1.7) and gKdV (1.10) also preserve the L^2 -norm of solutions. Thus, the Gaussian white noise μ_0 with the density:

$$d\mu_0 = Z^{-1} e^{-\frac{1}{2} \int_{\mathbb{T}} |\phi|^2 dx} d\phi \quad (1.15)$$

is expected to be invariant for these equations. On \mathbb{T} , a typical element ϕ in the support of the white noise μ_0 is represented by (1.4) (in the mean-zero case), which is almost surely in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$ for any $\varepsilon > 0$ but not in $H^{-\frac{1}{2}}(\mathbb{T})$. It is this low regularity that makes it difficult to rigorously study invariance of the white noise. Nonetheless, for KdV (1.10) with $p = 3$, the (mean-zero) white noise is shown to be invariant [41, 34, 37, 39]. See also [35, 38]. In particular, this result yields almost sure global existence for KdV with the

white noise as initial data, namely with $u|_{t=0} = \phi$, where ϕ is as in (1.4) conditioned that $g_{-n} = \overline{g_n}$.

Note that almost sure existence of a solution with the white noise as initial data (but not its invariance) also follows from deterministic global well-posedness of KdV in $H^{-1}(\mathbb{T})$ by Kappeler-Topalov [27], exploiting the integrable structure of the equation. However, the result in [34, 37] can be applied to non-integrable variants of KdV, and moreover it asserts a stronger form of uniqueness.

In [39], the white noise was shown to be a weak limit of invariant measures, more precisely, a limit of interpolations of the Gibbs measures (with a parameter) and the white noise. This result holds not only for KdV but also for cubic NLS and mKdV, i.e. (1.7) and (1.10) with $p = 4$. Due to lack of well-defined dynamics in the support of the white noise, this does not yield invariance of the white noise for cubic NLS and mKdV, but it only provides a strong evidence of such invariance.

1.3. Probabilistic Cauchy theory. In an effort to study the Cauchy problem for cubic NLS in low regularity, Colliander-Oh [17] considered the following *Wick ordered cubic NLS* on \mathbb{T} :

$$iu_t - u_{xx} \pm u(|u|^2 - 2 \int |u|^2 dx) = 0, \quad (1.16)$$

with random initial data of the form (1.3), where $\int |u|^2 dx := \frac{1}{2\pi} \int |u|^2 dx$. This equation first appeared in [7] in the context of the defocusing cubic NLS on \mathbb{T}^2 as an equivalent formulation of the Hamiltonian equation arising from the Wick ordered Hamiltonian.

Note that u solves (1.7) if and only if $v(t) = e^{i\gamma t}u(t)$, with $\gamma \in \mathbb{R}$, solves $i\partial_t v - v_{xx} \pm |v|^2 v + \gamma v = 0$. Hence, by letting $\gamma = \mp 2 \int |u|^2 dx$ along with the L^2 -conservation, (1.7) is equivalent to (1.16), at least for $u_0 \in L^2(\mathbb{T})$. For $u_0 \notin L^2(\mathbb{T})$, we cannot freely convert solutions of (1.16) into solutions of (1.7). See [40] for more discussions on the relation between the cubic NLS (1.7) and the Wick ordered cubic NLS (1.16).

In [17], it is shown that (1.16) is almost surely locally well-posed with the initial data (1.3) with $\alpha > \frac{1}{6}$, corresponding to $H^s(\mathbb{T})$, $s > -\frac{1}{3}$, and almost surely globally well-posed with the initial data (1.3) with $\alpha > \frac{5}{12}$, corresponding to $H^s(\mathbb{T})$, $s > -\frac{1}{12}$. Note that ϕ in (1.3) represents a typical element of the following Gaussian measure ρ_α with the density:

$$d\rho_\alpha = Z e^{-\frac{1}{2} \int_{\mathbb{T}} |\phi|^2 dx - \frac{1}{2} \int_{\mathbb{T}} |D^\alpha \phi|^2 dx} d\phi. \quad (1.17)$$

The probabilistic local argument in [17] closely follows that by Bourgain [7]. The main ingredients are (i) an improvement of the Strichartz estimates under randomization of initial data and (ii) hypercontractivity of the Ornstein-Uhlenbeck semigroup. Burq-Tzvetkov [13] exploited (i) to establish an almost sure local existence result for the nonlinear wave equation (NLW) for a wider class of randomizations.

The probabilistic global argument in [17] was the first almost sure global existence argument in the absence of conservation laws or formally invariant measures. The proof was based on the adaptation of Bourgain's high-low method [5] in the probabilistic setting. In particular, it exploited the L^2 -conservation and invariance of the Gaussian measure ρ_α in (1.17) under the *linear* flow. More recently, there have been almost sure global existence results for some other equations [15, 32, 11] in the absence of conservation laws or formally invariant measures. The argument is based on a combination of a conservation law at a higher regularity and a probabilistic argument.

In the following, we focus on the almost sure global existence result in [17]. It says that given $\alpha > \frac{5}{12}$, there exists $\Sigma_0 = \Sigma_0(\alpha)$ with $\rho_\alpha(\Sigma_0) = 1$ such that, if $\phi \in \Sigma_0$, then there exists a global solution u to (1.16) with $u|_{t=0} = \phi$. There are two issues about this almost sure global existence result:

- It does not say anything about what happens to Σ_0 under the dynamics. In particular, it does *not* guarantee that Σ_0 remains a set of full measure under the (1.16) flow. Let $\Phi(t) : \phi \mapsto u(t) = \Phi(t)\phi$ be the solution map of (1.16). Then, it may happen that $\Phi(t)\Sigma_0$ for $t > 0$ is a set of smaller measure and we may even have $\rho_\alpha(\Phi(t)\Sigma_0) = 0$ for some $t > 0$.
- The uniqueness statement for the local result in [17] states the following; if $\phi = \phi(\omega)$ is a “good” initial condition, then the solution $u(t) = \Phi(t)\phi$ exists up to time $\delta > 0$ and uniqueness holds in the ball centered at $S(t)\phi$ of radius 1 in $X_{[0,\delta]}^{0,\frac{1}{2}+}$. Here, $S(t) = e^{-it\partial_x^2}$ denotes the linear propagator for (1.16) and $X_{[0,\delta]}^{0,\frac{1}{2}+}$ denotes the local-in-time version of the $X^{s,b}$ space onto the time interval $[0, \delta]$ (with $s = 0$ and $b = \frac{1}{2}+$). This is a typical uniqueness statement for the probabilistic local Cauchy theory. See [7, 13]. However, the uniqueness statement for the almost sure global existence result in [17] holds in a much milder sense. See Remark 1.2 in [17].

The next theorem addresses both of the issues described above.

Theorem 1.2. *Let $\alpha > \frac{5}{12}$. Then, there exists a set $\Sigma \subset H^{\alpha-\frac{1}{2}-\varepsilon}(\mathbb{T})$, $\varepsilon > 0$, of full measure with respect to ρ_α such that*

- (i) *Σ is invariant under the (1.16)-dynamics. In particular, $\rho_\alpha(\Phi(t)\Sigma) = 1$ for any $t \in \mathbb{R}$ and if $\phi \in \Sigma$, then the corresponding solution $u(t) = \Phi(t)\phi$ exists globally.*
- (ii) *Given $\phi \in \Sigma$, the global solution $u(t) = \Phi(t)\phi$ is unique in the following sense. Given $t_* \in \mathbb{R}$, there exists positive $\delta = \delta(\phi, t_*) > 0$ such that uniqueness holds in the ball centered at $S(\cdot - t_*)u(t_*)$ of radius 1 in $X_{[t_*-\delta, t_*+\delta]}^{0, \frac{1}{2}+}$. Moreover, for each finite time interval I , $\delta > 0$ is bounded away from 0 for all $t_* \in I$.*

We point out that this uniqueness statement is in the spirit of the usual probabilistic local Cauchy theory and is stronger than the uniqueness statement for almost sure global solutions in [17]. We present the proof of Theorem 1.2 in Section 2. Previously, Burq-Tzvetkov [15] constructed an invariant set of full measure in considering almost sure global existence for NLW on \mathbb{T}^3 . Their idea was based on first characterizing the set of initial data such that the corresponding linear solutions satisfy some space-time bounds, guaranteeing global existence, and then showing that random initial data almost surely belongs to this set. The global argument in [17] exploits finer properties of *products* of the linear solutions with random initial data (such as the hypercontractivity of the Ornstein-Uhlenbeck semigroup), which is difficult to characterize in terms of individual initial data. Hence, the proof of Theorem 1.2 follows a different path than that in [15].

1.4. Cameron-Martin Theorem and almost sure global existence.

In this section, we recall Cameron-Martin Theorem and discuss its implications in the context of almost sure global existence for nonlinear Hamiltonian PDEs.

For this purpose, we first need to briefly go over the definition of abstract Wiener spaces introduced by Gross [24]. See also Kuo [28]. Let H be a real separable Hilbert space. It is known that the Gauss measure ρ with the density $d\rho = Z^{-1}e^{-\frac{1}{2}\|x\|_H^2}dx$ is only finitely additive if $\dim H = \infty$.

Let \mathcal{P} denotes the collection of all finite dimensional orthogonal projections of H . A seminorm $||| \cdot |||$ on H is said to be measurable if, for any $\varepsilon > 0$, there exists $\mathbb{P}_\varepsilon \in \mathcal{P}$ such that $\rho(|||\mathbb{P}x||| > \varepsilon) < \varepsilon$ for all $\mathbb{P} \in \mathcal{P}$ with $\mathbb{P} \perp \mathbb{P}_\varepsilon$. Let B be the completion of H with respect to this seminorm $||| \cdot |||$. Then, Gross [24] showed that ρ can be made sense of as a countably additive Gaussian measure on B . In this case, we say that the triplet (B, H, ρ) is an

abstract Wiener space. The original Hilbert space H is often referred to as a *Cameron-Martin space* or a reproducing kernel Hilbert space.

Let (B, H, ρ) be an abstract Wiener space. Then, Cameron-Martin Theorem states the following.

Cameron-Martin Theorem. *Given $h \in B$, define a shifted measure ρ_h by $\rho_h(\cdot) := \rho(\cdot - h)$. Then, the shifted measure ρ_h is mutually absolutely continuous with respect to ρ if and only if $h \in H$.*

This theorem also provides a precise expression of the Radon-Nikodym derivative. This absolute continuity under a shift in the direction of H leads to the H -differentiation, which plays a key role in the Malliavin Calculus. See Shigekawa [43].

Example 1. Consider the Wiener measure ρ in (1.12). More precisely, consider the Gaussian measure ρ with the density:

$$d\rho = Z^{-1} e^{-\frac{1}{2} \int_{\mathbb{T}} |\phi|^2 dx - \frac{1}{2} \int_{\mathbb{T}} |\phi_x|^2 dx} d\phi = Z^{-1} e^{-\frac{1}{2} \|\phi\|_{H^1}^2} d\phi. \quad (1.18)$$

Then, ρ is the Gauss measure on $H = H^1(\mathbb{T})$. It is known that, with $B = H^s(\mathbb{T})$, $s < \frac{1}{2}$, the triplet (B, H, ρ) is an abstract Wiener space. See Bényi-Oh [1] for examples of other Banach spaces B such that (B, H, ρ) is an abstract Wiener space. Note that (i) this Gaussian measure ρ is absolutely continuous with respect to the Gibbs measure μ in the defocusing case, i.e. with the minus sign in (1.9) and (ii) the Fourier-Wiener series (1.3) represents functions in the support of ρ . Then, as a corollary to invariance of the Gibbs measure μ and Cameron-Martin Theorem, we have the following statement.

Theorem 1.3. *Let $v_0 \in H^1(\mathbb{T})$. Then, the solution $u = u(x, t; \omega)$ to the defocusing NLS (1.7) with the initial data of the form*

$$u_0(x; \omega) = v_0(x) + \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle} e^{inx} \in H^{\frac{1}{2}-}(\mathbb{T}) \setminus H^{\frac{1}{2}}(\mathbb{T}), \text{ a.s.,}$$

exists globally in time, almost surely in ω .

For the cubic and quintic NLS ($p = 4$ and $p = 6$, respectively), Theorem 1.3 follows from the deterministic global well-posedness results by Bourgain [2, 10]. For $p > 6$, Theorem 1.3 does not follow from known (deterministic) results.

In the focusing case, the Gibbs measure μ comes with an L^2 -cutoff $\mathbf{1}_{\{\|\phi\|_{L^2} \leq B\}}$. Since a shift by v_0 does not preserve this L^2 -cutoff, a result analogous to Theorem 1.3 does not hold as a corollary to Cameron-Martin Theorem. Recall, however, that the Gibbs measure makes sense only for

$p \leq 6$ in the focusing case (where B is sufficiently small if $p = 6$), where deterministic global well-posedness results are available in the regularity of the Gibbs measures. When $p = 4$, the cubic NLS (1.7) is globally well-posed in $L^2(\mathbb{T})$. When $p = 6$, a modification of the argument in [10] yield global well-posedness of the quintic NLS for data with small L^2 -norms.

Example 2. In this example, we assume that all the functions are real-valued with mean zero on \mathbb{T} . Let μ_0 be the mean-zero Gaussian white noise defined in (1.15). Note that μ_0 is the Gauss measure on $H = L_0^2(\mathbb{T})$, where $L_0^2(\mathbb{T})$ denotes the collection of real-valued functions in $L^2(\mathbb{T})$ with mean zero on \mathbb{T} . With $B = H^s(\mathbb{T})$, $s < -\frac{1}{2}$, the triplet (B, H, μ_0) forms an abstract Wiener space. Hence, as a corollary to invariance of the white noise for KdV (1.10) with $p = 3$ and Cameron-Martin Theorem, we have the following.

Theorem 1.4. *Let $v_0 \in L_0^2(\mathbb{T})$. Then, the solution $u = u(x, t; \omega)$ to KdV (1.10) with $p = 3$ with the initial data of the form*

$$u_0(x; \omega) = v_0(x) + \sum_{n \in \mathbb{Z} \setminus \{0\}} g_n(\omega) e^{inx} \in H_0^{-\frac{1}{2}-}(\mathbb{T}) \setminus H_0^{-\frac{1}{2}}(\mathbb{T}), \text{ a.s.,}$$

exists globally in time, almost surely in ω . Here, $H_0^s(\mathbb{T})$ denotes the collection of real-valued functions in $H^s(\mathbb{T})$ with mean zero on \mathbb{T} .

Note that Theorem 1.4 also follows from the deterministic global well-posedness results by Kappeler-Topalov [27]. However, the result in [27] is not applicable to non-integrable variants of KdV, while Theorem 1.4 also holds for some non-integrable variants of KdV.

Example 3. The Gaussian measure ρ_α defined in (1.17) is the Gauss measure on $H = H^\alpha(\mathbb{T})$. With $B = H^s(\mathbb{T})$, $s < \alpha - \frac{1}{2}$, the triplet (B, H, ρ_α) forms an abstract Wiener space. Hence, as a corollary to Theorem 2 in [17] and Cameron-Martin Theorem, we have the following.

Theorem 1.5. *Let $v_0 \in H^\alpha(\mathbb{T})$, $\alpha > \frac{5}{12}$. Then, the solution $u = u(x, t; \omega)$ to the Wick ordered cubic NLS (1.16) with the initial data of the form*

$$u_0(x; \omega) = v_0(x) + \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle |n|^\alpha \rangle} e^{inx} \in H^{\alpha-\frac{1}{2}-}(\mathbb{T}) \setminus H^{\alpha-\frac{1}{2}}(\mathbb{T}), \text{ a.s.,} \quad (1.19)$$

exists globally in time, almost surely in ω .

It follows from a slight modification of the proof of Theorem 1 in [17] that the solution u to (1.16) with initial data (1.19), $\alpha > \frac{1}{3}$, exists locally in time, almost surely in ω , even if v_0 is only in $L^2(\mathbb{T})$. However, it seems that much more effort is required to modify the global argument in [17] to obtain Theorem 1.5 for $v_0 \in L^2(\mathbb{T})$.

1.5. On absolute continuity under a shift for other classes of randomizations. There are several results on almost sure global existence for a more general class of randomized initial data. See [15, 32, 11]. In this subsection, we discuss the effect of a shift by a smooth function on such randomized initial data. For simplicity, we restrict our attention to \mathbb{T}^d . Fix a function $u = \sum_{n \in \mathbb{Z}^d} \widehat{u}_n e^{in \cdot x}$ in $H^s(\mathbb{T}^d)$, and define its randomization u^ω by

$$u^\omega = \sum_{n \in \mathbb{Z}^d} a_n(\omega) \widehat{u}_n e^{in \cdot x},$$

where $\{a_n\}_{n \in \mathbb{Z}^d}$ is a sequence of independent complex-valued random variables on a probability space (Ω, \mathcal{F}, P) . Given a deterministic function v on \mathbb{T}^d , we consider a shifted function

$$w^\omega = v + u^\omega = \sum_{n \in \mathbb{Z}^d} (\widehat{v}_n + a_n(\omega) \widehat{u}_n) e^{in \cdot x}.$$

We would like to know when the distribution of w^ω is absolutely continuous with respect to that of u^ω . Clearly, the support of $a_n(\omega) \widehat{u}_n$ must contain the support of $\widehat{v}_n + a_n(\omega) \widehat{u}_n$. This eliminates a certain class of random variables such as the Bernoulli random variables. Moreover, since our interest is to determine a class of functions v such that the distribution of the shifted random function w^ω is absolutely continuous with respect to that of u^ω , we may assume that $\{a_n(\omega) \widehat{u}_n\}_{\omega \in \Omega} = \mathbb{C}$ for each $n \in \mathbb{Z}^d$. For simplicity, we set $\{a_n\}_{n \in \mathbb{Z}^d}$ to be a sequence of independent standard complex-valued Gaussian random variables $\{g_n\}_{n \in \mathbb{Z}^d}$ in the following.

Recall the following definition of the Hellinger integral. See, for example, Da Prato [18]. Given two probability measures μ and ν , the Hellinger integral of μ and ν is defined by

$$H(\mu, \nu) = \int_{\Omega} \sqrt{\frac{d\mu}{d\zeta} \frac{d\nu}{d\zeta}} d\zeta,$$

where $\zeta = \frac{1}{2}(\mu + \nu)$. Note that μ and ν are absolutely continuous with respect to ζ , so the Radon-Nikodym derivatives $\frac{d\mu}{d\zeta}$ and $\frac{d\nu}{d\zeta}$ make sense. If ν is absolutely continuous with respect to μ , we can write the Hellinger integral as

$$H(\mu, \nu) = \int_{\Omega} \sqrt{\frac{d\nu}{d\mu}} d\mu.$$

Given $\{\mu_n\}_{n \in \mathbb{N}}$ and $\{\nu_n\}_{n \in \mathbb{N}}$ are sequences of probability measures on \mathbb{C} , consider the product measures on \mathbb{C}^∞ : $\mu = \bigotimes_{n=1}^{\infty} \mu_n$ and $\nu = \bigotimes_{n=1}^{\infty} \nu_n$. In this case, the Hellinger integral of μ and ν is given by $H(\mu, \nu) = \prod_{n=1}^{\infty} H(\mu_n, \nu_n)$.

Then, Kakutani's theorem [26] states that (i) μ and ν are equivalent if $H(\mu, \nu) > 0$, and (ii) μ and ν are singular if $H(\mu, \nu) = 0$.

Now, let μ_n and ν_n be the probability measures on \mathbb{C} induced by $\omega \mapsto g_n(\omega)\hat{u}_n$ and $\omega \mapsto \hat{v}_n + g_n(\omega)\hat{u}_n$, respectively. Namely, the density functions are given by

$$d\mu_n = \frac{1}{2\pi} e^{-\frac{1}{2} \frac{|z|^2}{|\hat{u}_n|^2}} dz \quad \text{and} \quad d\nu_n = \frac{1}{2\pi} e^{-\frac{1}{2} \frac{|z - \hat{v}_n|^2}{|\hat{u}_n|^2}} dz = e^{-\frac{1}{2} \frac{|\hat{v}_n|^2 - \operatorname{Re}(\hat{v}_n, z)}{|\hat{u}_n|^2}} d\mu_n.$$

Then, the Hellinger integral of μ_n and ν_n is given by $H(\mu_n, \nu_n) = e^{-\frac{1}{8} \frac{|\hat{v}_n|^2}{|\hat{u}_n|^2}}$. Let $\mu = \bigotimes_{n \in \mathbb{Z}^d}^\infty \mu_n$ and $\nu = \bigotimes_{n \in \mathbb{Z}^d}^\infty \nu_n$. Then, μ and ν represent the probability distributions of (the Fourier coefficients of) u^ω and w^ω , respectively. Moreover, we have

$$H(\mu, \nu) = \prod_{n \in \mathbb{Z}^d} e^{-\frac{1}{8} \frac{|\hat{v}_n|^2}{|\hat{u}_n|^2}}.$$

Hence, μ and ν are equivalent if and only if $H(\mu, \nu)^{-1} < \infty$, i.e.

$$\sum_{n \in \mathbb{Z}^d} \frac{|\hat{v}_n|^2}{|\hat{u}_n|^2} < \infty. \quad (1.20)$$

In particular, if $\hat{u}_n = 0$ for some n , we must have $\hat{v}_n = 0$.

In general, given $u \in H^s(\mathbb{T}^d)$, it may not be easy to determine a class of functions v such that (1.20) holds. For example, suppose that $\hat{u}_n = |n|^{-1}$, $n \in \mathbb{Z}^d \setminus \{0\}$, and $\hat{u}_0 = 0$. Namely, u^ω is the mean-zero Gaussian free field on \mathbb{T}^d . In this case, we have $u \in \dot{H}^s(\mathbb{T}^d) \setminus \dot{H}^{1-\frac{d}{2}}(\mathbb{T}^d)$, $s < 1 - \frac{d}{2}$, almost surely. Since the randomization on the Fourier coefficients does not introduce any smoothing in terms of differentiability almost surely in ω , we also have $u^\omega \in \dot{H}^s(\mathbb{T}^d) \setminus \dot{H}^{1-\frac{d}{2}}(\mathbb{T}^d)$, $s < 1 - \frac{d}{2}$, almost surely. In view of the condition (1.20), we see that the distributions of u^ω and the shifted random variable w^ω are equivalent if $v \in \dot{H}^1(\mathbb{T}^d)$, and that they are singular if $v \notin \dot{H}^1(\mathbb{T}^d)$. If one knows what kind of noise u^ω is added to smooth initial data v , then it is possible to repeat the computation above.

1.6. On the large deviation principle with respect to small random perturbations. In this subsection, we discuss the large deviation principle for solutions with initial data perturbed by small random noises. In particular, we consider initial data of the form:

$$u_0^\varepsilon(x; \omega) = v_0(x) + \varepsilon \phi(x; \omega) \quad (1.21)$$

for small $\varepsilon > 0$, where v_0 is a deterministic smooth function and $\phi(\omega)$ is as in (1.2), (1.3), or (1.4). The theory of large deviations was formalized in

the seminal paper by Varadhan [47] and we follow his definition. See also Varadhan [48].

In the following, we consider KdV, (1.10) with $p = 3$, and use the notations in Theorem 1.4. Fix $v_0 \in L_0^2(\mathbb{T})$ and let $\phi(\omega)$ be the mean-zero Gaussian white noise given by (1.4). For these v_0 and $\phi(\omega)$, let $u^\varepsilon(\omega)$ be the global solution to KdV with initial data $u_0^\varepsilon(\omega)$ defined in (1.21). Note that Theorem 1.4 guarantees global existence of such u^ε almost surely. Fix $s < -\frac{1}{2}$. Then, the map $\omega \mapsto u^\varepsilon(\omega)$ induces probability measures μ_ε on $C(\mathbb{R}; H_0^s(\mathbb{T}))$. In the following, we discuss the large deviation principle for μ_ε .

First, we discuss the large deviation principle for the probability measures $\rho_\varepsilon = P \circ (u_0^\varepsilon)^{-1}$ on initial data u_0^ε in (1.21). Define a rate function $I : H_0^s(\mathbb{T}) \rightarrow [0, \infty]$ by

$$I(f) = \frac{1}{2} \|f - v_0\|_{L_0^2(\mathbb{T})}^2. \quad (1.22)$$

Note that (i) I is lower semicontinuous by Fatou's lemma and (ii) $K_r = \{f \in H_0^s(\mathbb{T}) : I(f) \leq r\} \subset L_0^2(\mathbb{T})$ is compact in $H_0^s(\mathbb{T})$ for each finite $r \geq 0$. Then, the large deviation principle holds for $\{\rho_\varepsilon\}_{\varepsilon>0}$ with the rate function I in (1.22). Namely, we have

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log \rho_\varepsilon(F) \leq - \inf_{f \in F} I(f) \quad (1.23)$$

for any closed set $F \subset H_0^s(\mathbb{T})$ and

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log \rho_\varepsilon(G) \geq - \inf_{f \in G} I(f) \quad (1.24)$$

for any open set $G \subset H_0^s(\mathbb{T})$. The inequalities (1.23) and (1.24) follow from Theorems 3.3 and 4.2 in Chapter 3 of Freidlin-Wentzell [23]. Note that $K_0 = \{f \in H_0^s(\mathbb{T}) : I(f) = 0\}$ consists of a single function v_0 . Hence, it follows from Remark 2.3 in [48] that ρ_ε converges weakly to δ_{v_0} as $\varepsilon \rightarrow 0$.

Next, we discuss the large deviation principle for the probability measures μ_ε on solutions to KdV. First, for fixed $s \in [-1, -\frac{1}{2})$, endow $C(\mathbb{R} : H_0^s(\mathbb{T}))$ with the topology of compact convergence (compact-open topology) induced by the usual metric:

$$d(u, v) = \sum_{j=1}^{\infty} 2^{-j} \frac{\|u - v\|_{L_t^\infty([-j, j]; H^s)}}{1 + \|u - v\|_{L_t^\infty([-j, j]; H^s)}}.$$

Then, $C(\mathbb{R} : H_0^s(\mathbb{T}))$ is a Polish space. In view of global well-posedness of KdV in $H_0^{-1}(\mathbb{T})$ by Kappeler-Topalov [27], let $X \subset C(\mathbb{R} : H_0^s(\mathbb{T}))$ denote the collection of global-in-time solutions to KdV constructed in [27], endowed with the subspace topology. It follows from the continuity of the solution

map $\Phi : u(0) \in H_0^s(\mathbb{T}) \mapsto \Phi(u(0)) := u \in X$ with respect to the topology induced by the metric $d(\cdot, \cdot)$ that X is also a Polish space.

Let $\Psi := \Phi^{-1} : X \rightarrow H_0^s(\mathbb{T})$ be the evaluation map given by $\Psi(u) = u(0)$. By definition of μ_ε , we have $\mu_\varepsilon(A) = \rho_\varepsilon(\Psi(A))$. Now, define a rate function $\tilde{I} : X \rightarrow [0, \infty]$ by

$$\tilde{I}(u) := I(\Psi(u)) = \frac{1}{2} \|\Psi(u) - v_0\|_{L_0^2(\mathbb{T})}^2. \quad (1.25)$$

The lower semicontinuity of \tilde{I} directly follows from that of I . Let $\tilde{K}_r = \{u \in X : \tilde{I}(u) \leq r\} \subset C(\mathbb{R} : L_0^2(\mathbb{T}))$. Given a sequence $\{u_n\}_{n=1}^\infty \subset \tilde{K}_r$, it follows from (1.25) that $\{u_n(0)\}_{n=1}^\infty$ is bounded in $L_0^2(\mathbb{T})$ and thus is precompact in $H_0^s(\mathbb{T})$. Then, there exists a subsequence also denoted by $\{u_n\}_{n=1}^\infty$ such that $u_n(0)$ converges to $u_\infty(0)$ in $H_0^s(\mathbb{T})$. By the continuity of the solution map $u(0) \in H_0^s(\mathbb{T}) \mapsto u \in X$, u_n converges to $u_\infty := \Phi(u_\infty(0))$ in X . By weak convergence in $L_0^2(\mathbb{T})$ of (a further subsequence of) $\{u_n\}_{n=1}^\infty$, it is easy to see that $u_\infty \in \tilde{K}_r$. This shows that \tilde{K}_r is compact in X .

By the continuity of the solution map $\Phi : H_0^s(\mathbb{T}) \rightarrow X$, we see that $\Psi(F) = \Phi^{-1}(F)$ is closed (and open) in $H_0^s(\mathbb{T})$ if F is closed (and open, respectively) in X . Hence, as a direct consequence of (1.23) and (1.24), we have the following large deviation principle for $\{\mu_\varepsilon\}_{\varepsilon>0}$ with the rate function \tilde{I} defined in (1.25). We have

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mu_\varepsilon(F) \leq - \inf_{u \in F} \tilde{I}(u) \quad (1.26)$$

for any closed set $F \subset X$ and

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mu_\varepsilon(G) \geq - \inf_{u \in G} \tilde{I}(u) \quad (1.27)$$

for any open set $G \subset X$. Let v be the unique solution to KdV such that $v(0) = v_0$. Since $\tilde{K}_0 := \{u \in X : \tilde{I}(u) = 0\}$ consists of a single function v , it follows again from Remark 2.3 in [48] that μ_ε converges weakly to δ_v .

Remark 1.6. Due to lack of a good well-posedness theory below $L^2(\mathbb{T})$, the large deviation principle for the Wick ordered cubic NLS (1.16) holds only in some mild sense. Fix $v_0 \in H^\alpha(\mathbb{T})$ with $\alpha > \frac{5}{12}$ and let $\phi(\omega)$ as in (1.3). With $s < \alpha - \frac{1}{2}$, define a rate function $I : H^s(\mathbb{T}) \rightarrow [0, \infty]$ by

$$I(f) = \frac{1}{2} \|f - v_0\|_{H^\alpha(\mathbb{T})}^2. \quad (1.28)$$

Then, with $\rho_\varepsilon = P \circ (u_0^\varepsilon)^{-1}$, the large deviation principle, i.e. (1.23) and (1.24), holds for $\{\rho_\varepsilon\}_{\varepsilon>0}$ as before.

However, the large deviation principle for the probability measures μ_ε induced by $\omega \mapsto u^\varepsilon(\omega)$ holds only in a weak sense. Let X denote the collection of all (known) solutions to (1.16) with initial data in $H^s(\mathbb{T})$. Due

to lack of a good well-posedness theory below $L^2(\mathbb{T})$, we do not know if X is a Polish space and thus we can only draw a weak conclusion for μ_ε . With the previous notations, we can define a rate function $\tilde{I}(u) := I(\Psi(u)) = I(u(0))$, where I is as in (1.28). Then, we have the following ‘weak’ large deviation principle for $\{\mu_\varepsilon\}_{\varepsilon>0}$. Namely, (1.26) holds if $\Psi(F)$ is closed in $H^s(\mathbb{T})$, while (1.27) holds if $\Psi(G)$ is open in $H^s(\mathbb{T})$. Lastly, since $\tilde{K}_0 := \{u \in X : \tilde{I}(u) = 0\}$ consists of a single function v , where v is the unique solution to (1.16) such that $v(0) = v_0$, we would like to conclude that μ_ε converges weakly to δ_v . Such weak convergence, however, does not follow at this point due to lack of continuous dependence on $H^s(\mathbb{T})$ of the solution map to (1.16) constructed in [17] and Theorem 1.2 above.

2. INVARIANT SET OF FULL MEASURE FOR ALMOST SURE GLOBAL EXISTENCE OF THE WICK ORDERED CUBIC NLS

In this section, we present the proof of Theorem 1.2. Let $\Phi(t) : u(0) \mapsto u(t)$ be the solution map of (1.16), sending an initial condition $u(0)$ to the solution $u(t)$ at time t , and $S(t) = e^{-it\partial_x^2}$ be the linear propagator for (1.16). In the following, we fix $\alpha \in (\frac{5}{12}, \frac{1}{2}]$ and $s = \alpha - \frac{1}{2} - \varepsilon < 0$, $\varepsilon > 0$. The almost sure global result in [17] states that there exists a set $\Sigma_0 = \Sigma_0(\alpha)$ with

$$\rho_\alpha(\Sigma_0) = 1 \quad (2.1)$$

such that if $\phi = \phi(\omega) \in \Sigma_0$, the corresponding solution $u(t) = \Phi(t)\phi$ exists globally. Here, ρ_α is as in (1.17) and $\phi \in \Sigma_0$ can be represented by (1.3) almost surely. Note that ϕ is almost surely in $H^s(\mathbb{T}) \setminus H^{\alpha-\frac{1}{2}}(\mathbb{T})$. In particular, it is not in $L^2(\mathbb{T})$ almost surely for $\alpha < \frac{1}{2}$. In establishing this result, we exhibited nonlinear smoothing under randomization of the initial data, i.e. if $\phi \in \Sigma_0$, then, although the linear solution $S(t)\phi$ is not in $L^2(\mathbb{T})$ for any $t \in \mathbb{R}$ almost surely, the nonlinear part $v(t) := \Phi(t)\phi - S(t)\phi$ of the solution is in $L^2(\mathbb{T})$ for each $t \in \mathbb{R}$. Once we restrict our attention to the local-in-time setting, we know more properties about this flow. We summarize the local-in-time properties of the flow. See [17, Theorem 1].

Proposition 2.1 (Summary of the local result in [17]). *Fix $\delta \ll 1$. Then, there exists $\Omega_\delta \in \mathcal{F}$ with the following properties.*

(i) *The complementary measure of Ω_δ is small. More precisely, we have*

$$P(\Omega_\delta^c) < e^{-\frac{1}{\delta^\varepsilon}}. \quad (2.2)$$

(ii) *For each $\omega \in \Omega_\delta$, there exists a unique solution u to (1.16) in*

$$S(t)\phi(\omega) + C([- \delta, \delta]; L^2(\mathbb{T})) \subset C([- \delta, \delta]; H^s(\mathbb{T}))$$

with initial condition $u|_{t=0} = \phi(\omega)$, where $\phi(\omega)$ is given by (1.3).

(iii) Let $\omega \in \Omega_\delta$ and $u(t) = \Phi(t)\phi(\omega) = S(t)\phi(\omega) + v(t)$ be the solution to (1.16) constructed in (ii). Then, there exist C and $\theta > 0$ such that

$$\|v\|_{X_{[-\delta', \delta']}^{0, \frac{1}{2}+}} \leq C(\delta')^\theta \quad (2.3)$$

for $\delta' \leq \delta$. Here, $X_{[-\delta', \delta']}^{0, \frac{1}{2}+}$ denotes the local-in-time version of the $X^{\sigma, b}$ -space restricted onto the time interval $[-\delta', \delta']$ with $\sigma = 0$ and $b = \frac{1}{2}+$. In particular, we have

$$\sup_{t \in [-\delta', \delta']} \|v(t)\|_{L^2(\mathbb{T})} \leq C'(\delta')^\theta \quad (2.4)$$

for some $C' > 0$.

(iv) Let $w_0 \in L^2(\mathbb{T})$ with $\|w_0\|_{L^2} = m$. Then, there exists positive $\delta' = \delta'(m, \delta) < \delta$ such that, for each $\omega \in \Omega_\delta$, there exists a unique solution $u \in C([t_* - \delta', t_* + \delta']; H^s(\mathbb{T}))$ to (1.16) with initial condition $u|_{t=t_*} = S(t_*)\phi(\omega) + w_0$ as long as $[t_* - \delta', t_* + \delta'] \subset [-\delta, \delta]$. Here, uniqueness holds in the ball centered at $S(\cdot - t_*)(S(t_*)\phi + w_0)$ of radius 1 in $X_{[t_* - \delta', t_* + \delta']}^{0, \frac{1}{2}+}$.

While (i) and (ii) of Proposition 2.1 are exactly as in [17, Theorem 1], (iii) and (iv) follow directly from (a modification of) the proof of [17, Theorem 1]. In particular, (iv) holds since the required probabilistic estimates for the local argument in [17] hold uniformly for any subinterval $[t_* - \delta', t_* + \delta'] \subset [-\delta, \delta]$ if $\omega \in \Omega_\delta$. We point out that we do not know if an analogue of (iv) holds for the global-in-time setting. Namely, given $w_0 \in L^2$, (a small modification of) the proof of [17, Theorem 2] does not yield almost sure global existence for (1.16) with initial data $u|_{t=0} = \phi(\omega) + w_0$, where ϕ is as in (1.3).

In the following, we construct a set Σ of full measure, which is invariant under the (1.16)-dynamics. Moreover, our construction yields an enhanced uniqueness statement (see Theorem 1.2 (ii)).

• **Step 1:** First, we use the invariance of ρ_α under the *linear* flow and construct a set $\tilde{\Sigma}$ of full measure such that the linear solutions with initial data in $\tilde{\Sigma}$ have some desired property.

For small $\delta > 0$, let $\Sigma_\delta = \phi(\Omega_\delta)$, where $\phi : \Omega \rightarrow H^s(\mathbb{T})$ is the map given by (1.3) and Ω_δ is as in Proposition 2.1. Note that solutions with initial data in Σ_δ satisfy a good (local-in-time) uniqueness property, coming from the local argument in [17]. Letting

$$\tilde{\Sigma}_n := \Sigma_0 \cap \Sigma_{\frac{1}{n}},$$

we have $\rho_\alpha(\tilde{\Sigma}_n^c) < e^{-n^c}$ for $n \geq N$. Here, N is a sufficiently large integer such that Proposition 2.1 holds for all positive $\delta < N^{-1} \ll 1$. Then, define $\hat{\Sigma}_n$ by

$$\hat{\Sigma}_n := \bigcap_{k=0}^n S\left(-\frac{k}{n}\right)(\tilde{\Sigma}_n).$$

Since ρ_α is invariant under the linear flow, we have

$$\rho_\alpha(\hat{\Sigma}_n^c) < (n+1)e^{-n^c}. \quad (2.5)$$

Next, define $\tilde{\Sigma}_{[0,1]}$ by

$$\tilde{\Sigma}_{[0,1]} := \bigcup_{n=N}^{\infty} \hat{\Sigma}_n.$$

Note that $\rho_\alpha(\tilde{\Sigma}_{[0,1]}) = 1$, since (2.5) yields

$$\rho_\alpha((\tilde{\Sigma}_{[0,1]})^c) \leq \inf_{n \geq N} (n+1)e^{-n^c} = 0.$$

Finally, define $\tilde{\Sigma}$ by

$$\tilde{\Sigma} := \bigcap_{j \in \mathbb{Z}} S(-j)\tilde{\Sigma}_{[0,1]}.$$

Then, by the invariance of ρ_α under the linear flow, we have $\rho_\alpha(\tilde{\Sigma}) = 1$.

We claim that if $\phi \in \tilde{\Sigma}$, then given $t_* \in [j, j+1] \subset \mathbb{R}$, the conclusion of Proposition 2.1 (iv) holds with $\delta = \frac{1}{n}$ for some $n = n(\phi, j) \geq N$. More precisely, by writing $t_* = j + \tau$ with $\tau \in [0, 1]$, we have $S(t_*)\phi = S(\tau)\psi$ for some $\psi = S(j)\phi \in \tilde{\Sigma}_{[0,1]}$, i.e. $\psi \in \hat{\Sigma}_n$ for some $n = n(\phi, j)$. By further writing $\tau = \frac{k}{n} + \xi$ with $|\xi| < \frac{1}{2n}$ for some $k \in \{0, \dots, n\}$, we have $\varphi = S(\frac{k}{n})\psi \in \tilde{\Sigma}_n \subset \Sigma_{\frac{1}{n}}$. Then, by Proposition 2.1 (iv), given $w_0 \in L^2(\mathbb{T})$ with $\|w_0\|_{L^2} = m$, there exists $\delta' = \delta'(m, \delta) > 0$ (with $\delta = \frac{1}{n}$) such that a solution u to (1.16) with $u|_{t=\xi} = S(\xi)\varphi + w_0$ exists on $[\xi - \delta', \xi + \delta']$ and is unique in the ball centered at $S(\cdot)(S(\xi)\varphi + w_0)$ of radius 1 in $X_{[\xi - \delta', \xi + \delta']}^{0, \frac{1}{2}+}$. Here, we assumed that $\delta' < \frac{\delta}{2} = \frac{1}{2n}$ such that the subinterval $[\xi - \delta', \xi + \delta']$ lies in $[-\delta, \delta] = [-\frac{1}{n}, \frac{1}{n}]$. Finally, note that

$$S(\xi)\varphi = S(\tau)\psi = S(t_*)\phi.$$

Therefore, it follows from the discussion above that there exists a unique solution u to (1.16) on $[t_* - \delta', t_* + \delta']$ with $u|_{t=t_*} = S(t_*)\phi + w_0$, where uniqueness holds in the ball centered at $S(\cdot)(S(t_*)\phi + w_0)$ of radius 1 in $X_{[t_* - \delta', t_* + \delta']}^{0, \frac{1}{2}+}$.

• **Step 2:** Next, we construct the desired set Σ . Define Σ by

$$\Sigma = \bigcup_{t \in \mathbb{R}} \Phi(-t)(\Sigma_0 \cap \tilde{\Sigma}).$$

Recall that ρ_α is defined on the completion of the Borel σ -algebra on $H^s(\mathbb{T})$. See Remark 2.2 below. Since $\Sigma \supset \Sigma_0 \cap \tilde{\Sigma}$ with $\rho_\alpha(\Sigma_0 \cap \tilde{\Sigma}) = 1$, it follows that Σ is measurable and $\rho_\alpha(\Sigma) = 1$. By definition, the set Σ is invariant under the (1.16)-dynamics. Moreover, if $\phi \in \Sigma$, we have a global solution

$$u(t) = \Phi(t)\phi = S(t)\phi + v(t), \quad (2.6)$$

where $v(t) \in L^2(\mathbb{T})$ for each $t \in \mathbb{R}$.

Given $\phi \in \Sigma$, we have $\phi = \Phi(-t_0)\psi$ for some $t_0 \in \mathbb{R}$ and $\psi \in \Sigma_0 \cap \tilde{\Sigma}$. In particular, given $t_* \in [j, j+1] \subset \mathbb{R}$, we have

$$u(t_*) = \Phi(t_*)\phi = \Phi(t_* - t_0)\psi = S(t_* - t_0)\psi + w(t_*)$$

for some $w(t_*) \in L^2(\mathbb{T})$. Then, by Step 1, there exist $\delta = \delta(\phi, t_*) = \delta(\phi, j) > 0$ and $\delta' = \delta'(t_*) = \delta'(\|w(t_*)\|_{L^2}, \delta) > 0$ such that uniqueness holds in the ball of radius 1 centered at $S(\cdot)u(t_*)$ in $X_{[t_* - \delta', t_* + \delta']}^{0, \frac{1}{2}+}$. Lastly, given a finite time interval $I \subset [J_1, J_2] \subset \mathbb{R}$, we have $\inf_{t_* \in I} \delta'(t_*) > 0$ since $\sup_{t_* \in I} \|w(t_*)\|_{L^2} < \infty$ and $\inf_{t_* \in I} \delta(t_*) = \inf\{\delta(\phi, j) : j = J_1, J_1 + 1, \dots, J_2\} > 0$. This completes the proof of Theorem 1.2.

Remark 2.2. The Gaussian measure ρ_α is the induced probability measure under the map $\phi : \omega \in \Omega \mapsto \phi^\omega \in H^s(\mathbb{T})$, $s < \alpha - \frac{1}{2}$, defined in (1.3). In the following, we directly show that ρ_α is defined on the Borel σ -algebra in $H^s(\mathbb{T})$. Let ϕ_N be the Fourier truncation of ϕ given by

$$\phi_N(x; \omega) = \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle |n|^\alpha \rangle} e^{inx}. \quad (2.7)$$

Then, the set $A_{N,r} = \{\omega \in \Omega : \|\phi_N(\omega)\|_{H^s} \leq r\}$ is clearly measurable for each $N \in \mathbb{N}$ and for any $r \geq 0$. With $A_r = \{\omega \in \Omega : \|\phi(\omega)\|_{H^s} \leq r\}$, we have $A_r = \bigcap_{N \in \mathbb{N}} A_{N,r}$ and hence A_r is also measurable. Let $B_r(v)$ be the open ball of radius r centered at $v \in H^s(\mathbb{T})$. Then, by writing $B_r(v) = \bigcup_{n=1}^{\infty} \overline{B}_{r-\frac{1}{n}}(v)$, we see that $\phi^{-1}(B_r(v))$ is measurable. Since $H^s(\mathbb{T})$ is separable, any open set can be written as a countable union of open balls in $H^s(\mathbb{T})$. Hence, we conclude that ρ_α is defined on the Borel σ -algebra in $H^s(\mathbb{T})$ (and on its completion).

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